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OPTIMIZATION OF THE DAMPING DECREMENT OF FREE OSCILLATIONS OF A VISCOELASTIC LAYERED SPHERE WITH A LIMITATION ON WEIGHT

V. A. Budugaeva

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The problem of synthesis of a multilayer spherical shell with maximum damping of natural oscillations from a finite set of viscoelastic materials is considered with a limitation on weight. The necessary conditions of optimality are obtained, a computational algorithm is derived, and an example of calculation is presented.

1. Solving problems of the optimal design of layered structures, Bondarev et al. [1] showed that the damping level of a multilayer spherical shell made of a given set of viscoelastic materials can be increased by changing the relative position and relative thickness of the layers. An algorithm for optimal designing of shells of constant thickness with maximum damping of natural oscillations was developed in [1] using the conformity principle. It seems expedient to generalize the result obtained by introducing certain limitations on the working characteristics of the structure, for example, by limiting its weight.

We formulate the following problem (see [1]): from a given set of materials, it is required to construct a multilayer shell with limited weight that ensures a minimum value of the selected quality criterion:

$$F_0 = \mathrm{Im}\,[\omega(\theta)]. \tag{1.1}$$

Here $\omega = \omega_R + i\omega_I$ is the complex natural frequency and θ is the distribution of the density ρ and the Lamé parameters $\bar\lambda$ and $\bar\mu$ along the radial coordinate.

The natural frequency and form of free oscillations of the spherical shell are determined by the solution of the following problem:

$$\frac{\partial\sigma_r}{\partial r} + 2\,\frac{\sigma_r - \sigma_\varphi}{r} = \rho\,\frac{\partial^2 u}{\partial t^2}, \qquad \sigma_r = (\bar\lambda + 2\bar\mu)\,\frac{\partial u}{\partial r} + 2\bar\lambda\,\frac{u}{r},$$
$$\sigma_\varphi = 2(\bar\lambda + \bar\mu)\,\frac{u}{r} + \bar\lambda\,\frac{\partial u}{\partial r}, \qquad l < r < R, \qquad \sigma_r(l) = \sigma_r(R) = 0. \tag{1.2}$$

Here R is the outside radius of the shell, l is the inside radius of the shell, which is determined during solution of the problem, and the complex Lamé parameters have the form [2]

$$\bar\lambda_n = \lambda_n[1 - \Gamma^c_{\lambda n}(\omega_R) - \Gamma^s_{\lambda n}(\omega_R)], \qquad \bar\mu_n = \mu_n[1 - \Gamma^c_{\mu n}(\omega_R) - \Gamma^s_{\mu n}(\omega_R)],$$

where

$$\Gamma^c_{\lambda n}(\omega_R) = \int\limits_0^\infty R_{\lambda n}(\tau)\cos(\omega_R\tau)\,d\tau; \qquad \Gamma^s_{\lambda n}(\omega_R) = \int\limits_0^\infty R_{\lambda n}(\tau)\sin(\omega_R\tau)\,d\tau, \tag{1.3}$$

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$$\Gamma^c_{\mu n}(\omega_R) = \int\limits_0^\infty R_{\mu n}(\tau)\cos(\omega_R\tau)\,d\tau; \qquad \Gamma^s_{\mu n}(\omega_R) = \int\limits_0^\infty R_{\mu n}(\tau)\sin(\omega_R\tau)\,d\tau$$

[the subscript n is the layer number ($n = \overline{1, N}$)].

In calculating the integrals (1.3), we used relaxation kernel relations of the form [3] $R = \bar{a}\exp(-\beta t)/t^{1-\alpha}$, where $\bar{a}$, α, and β are empirical constants.

For a homogeneous shell, the complex frequency is found from the transcendental equation obtained by the solution of problem (1.2):

$$\cos[k(R-l)]\Big\{\frac{4\mu k}{l}\Big(\frac{4\mu}{R^2} - \rho\omega^2\Big) - \frac{4\mu k}{R}\Big(\frac{4\mu}{l} - \rho\omega^2\Big)\Big\}$$
$$+ \sin[k(R-l)]\Big\{\frac{(4\mu k)^2}{Rl} + \Big(\frac{4\mu}{l^2} - \rho\omega^2\Big)\Big(\frac{4\mu}{R^2} - \rho\omega^2\Big)\Big\} = 0. \tag{1.4}$$

Here $k^2 = \rho\omega^2/(\lambda + 2\mu)$ (the bar above λ and μ is omitted).

For a layered shell, the density and the Lamé parameters are piecewise-constant functions of the radius, and the displacement and stresses on the boundaries of the layers are continuous. This allows us to extend solution (1.4) to the case of a layered shell and to obtain the following recursive relation for the stresses and displacements:

$$\begin{pmatrix} u_N(R) \\ \sigma_N(R) \end{pmatrix} = G \begin{pmatrix} u_1(l) \\ \sigma_1(l) \end{pmatrix}. \tag{1.5}$$

In this case, the following boundary conditions are satisfied:

$$\sigma_N(R) = \sigma_1(l) = 0. \tag{1.6}$$

In relation (1.5), G is the resultant matrix of the form

$$G = \begin{Vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{Vmatrix} = \Lambda_n \cdot \Lambda_{n-1} \cdots \Lambda_1,$$

where

$$\Lambda_n = \Delta_n \begin{Vmatrix} a_n(r_n) & ib_n(r_n) \\ c_n(r_n) & id_n(r_n) \end{Vmatrix} \begin{Vmatrix} -d_n(r_{n-1}) & b_n(r_{n-1}) \\ -ic_n(r_{n-1}) & ia_n(r_{n-1}) \end{Vmatrix};$$

$$\Delta_n = (c_n(r_{n-1})b_n(r_{n-1}) - d_n(r_{n-1})a_n(r_{n-1}))^{-1} \qquad (n = \overline{1, N});$$

$$a_n = -\frac{k_n^2}{r}\cos(k_n r) + \frac{k_n}{r^2}\sin(k_n r); \qquad b_n = -\frac{k_n}{r^2}\cos(k_n r) - \frac{k_n^2}{r}\sin(k_n r);$$

$$c_n = \frac{k_n}{r}\Big[\frac{4\mu_n k_n}{r}\cos(k_n r) - \Big(\frac{4\mu_n}{r^2} - \rho_n\omega^2\Big)\sin(k_n r)\Big];$$

$$d_n = \frac{k_n}{r}\Big[\frac{4\mu_n k_n}{r}\sin(k_n r) + \Big(\frac{4\mu_n}{r^2} - \rho_n\omega^2\Big)\cos(k_n r)\Big], \quad r_{n-1} < r < r_n \quad (n = \overline{1, N}).$$

Solving (1.5) with allowance for conditions (1.6), we obtain the following secular equation for the frequency of a multilayer spherical shell:

$$g_{21} = 0. \tag{1.7}$$

2. We consider the problem of optimization of a viscoelastic, multilayer, hollow sphere that ensures maximum damping of free oscillations and has a limitation on weight. Using the representation $u(r,t) = \exp(i\omega t)z_1(r)$, $\sigma(r,t) = \exp(i\omega t)z_2(r)$, we reduce the initial problem (1.2) to solving a system of ordinary differential equations. Introducing the dimensionless variables $z_1 = u/r_0$, $z_2 = \sigma_r$, $r^* = r/r_0$, $l^* = l/r_0$

$\rho^* = \rho/\rho_0$, $\lambda^* = \lambda/\sigma_0$, $\mu^* = \mu/\sigma_0$, and $\omega^{*2} = \omega^2 r_0^2 \rho_0/\sigma_0$, where r_0, σ_0, and ρ_0 are the characteristic length, stress, and density, and transforming the variable segment of integration $[l, R]$ into the constant segment $[0, 1]$ by replacing

$$r = l + x(R - l), \qquad x \in [0, 1], \tag{2.1}$$

we obtain

$$Z^1 = AZ \equiv f, \qquad z_2(0) = z_2(1) = 0. \tag{2.2}$$

Here

$$A = \begin{Vmatrix} -\dfrac{2\lambda(R-l)}{r(\lambda+2\mu)} & \dfrac{R-l}{\lambda+2\mu} \\ \Big[\dfrac{4\mu(3\lambda+2\mu)}{r^2(\lambda+2\mu)} - \rho\omega^2\Big](R-l) & -\dfrac{4\mu(R-l)}{r(\lambda+2\mu)} \end{Vmatrix}; \qquad Z = \{z_1, z_2\}; \quad f = \{f_1, f_2\}.$$

We assume that the outside radius R is fixed and the inside radius l is determined during solution of the problem. As the control, we chose the pair $\{\theta(x), l\}$, where $\theta(x) = \{\rho(x), \lambda(x), \mu(x)\}$ is the distribution of the density and the Lamé parameters. The functional (1.1), which has the meaning of the damping decrement of natural oscillations of the sphere, is minimized with limitation on the weight of the sphere:

$$F_1(\rho, l) \equiv P_* - \eta P \leqslant 0. \tag{2.3}$$

Here P_* is the specified weight and η is a certain number.

The quantity $\omega^2(\theta, l)$ is found from system (2.2):

$$\omega^2(\theta, l) = \int_0^1 J_1(x, Z, \theta, l)\, dx \Big/ \int_0^1 J_2(x, Z, \theta, l)\, dx. \tag{2.4}$$

Here

$$J_1(x, Z, \theta, l) = \Big(\frac{4\mu(3\lambda+2\mu)}{r^2(\lambda+2\mu)} z_1^2 + \frac{1}{\lambda+2\mu} z_2^2\Big) r^2, \quad J_2(x, Z, \theta, l) = \rho z_1^2 r^2,$$

and r depends on x under (2.1). In this case, the main part of the increment of the functional (2.4) for the variations

$$\tilde{\theta}(x) = \begin{cases} t(x) = \{\rho(x), \lambda(x), \mu(x)\}, & x \in D, \quad t \in W, \\ \theta(x) = \{\tilde{\rho}(x), \tilde{\lambda}(x), \tilde{\mu}(x)\}, & x \notin D \end{cases} \tag{2.5}$$

($D \in [0, 1]$ is a set of small measure, $W = \{\theta_1, \ldots, \theta_m\}$ is a specified finite discrete set, and m is the number of different materials) has the form

$$\delta\omega^2 = \Big[\int_0^1 J_2(\,\cdot\,, \theta)\, dx\Big]^{-1} \Big\{ \int_D [H(\,\cdot\,, \tilde{\theta}) - H(\,\cdot\,, \theta)]\, dx + B\delta l \Big\}. \tag{2.6}$$

Here

$$H(\,\cdot\,, \tilde{\theta}) = \Big[J_1(\,\cdot\,, \tilde{\theta}) - \omega^2 J_2(\,\cdot\,, \tilde{\theta}) - \frac{2}{R-l} r^2 z_2 f_1(\,\cdot\,, \tilde{\theta})\Big]$$

$[H(\,\cdot\,, \theta)$ has a similar form$]$ and

$$B = \int_0^1 \Big[\frac{\partial J_1(\,\cdot\,, \theta)}{\partial l} - \omega \frac{\partial J_2(\,\cdot\,, \theta)}{\partial l} - \frac{2}{R-l} r^2 z_2 \frac{\partial f_1(\,\cdot\,, \theta)}{\partial l}\Big] dx;$$

the dot indicates the omitted arguments x, Z, and l.

From the limitation (2.3) it is possible to express δl, assuming that on variations (2.5), $\delta F_1 = 0$:

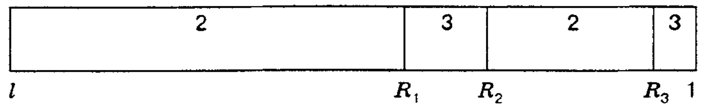

Fig. 1

$$\delta P = \int_D (R-l)(\tilde{\rho}(x) - \rho(x))r^2\,dx + B_1\delta l = 0. \tag{2.7}$$

Here $B_1 = \int_0^1 \rho(x)r[(R-l)(2-3x) - l]\,dx$. Substituting (2.7) into (2.6), we finally obtain

$$\delta F_0 = \delta\,\mathrm{Im}\,(\omega) = \int_D \mathrm{Im}\Big\{\Big[\int_0^1 J_2(\,\cdot\,,\theta)\,dx\Big]^{-1}(2\omega)^{-1}[T(\,\cdot\,,\tilde{\theta}) - T(\,\cdot\,,\theta)]\Big\}dx,$$

where $T(\,\cdot\,,\tilde{\theta}) = H(\,\cdot\,,\tilde{\theta}) - (B/B_1)(R-l)r^2\tilde{\rho}(x)$.

The minimizing sequence of the control $[\theta(x), l]$ is constructed using the negative formulation of Pontryagin's principle of maximum (see [4]), from which it follows that in the neighborhood of the nonoptimal control $[\theta(x), l]$, a new control $[\theta(x), l+\delta l]$ exists, which improves the structure, by satisfying the limitation (2.3). This control is sought from the minimum condition for δF_0, i.e,

$$\int_{D_i} \mathrm{Im}\Big[\Big[\int_0^1 J_2(\cdot,\theta)\,dx\Big]^{-1}(2\omega)^{-1}T(\cdot,\tilde{\theta})\Big]dx = \min_{\theta\in W}\int_{D_i}\mathrm{Im}\Big[\Big[\int_0^1 J_2(\cdot,\theta)\,dx\Big]^{-1}(2\omega)^{-1}T(\cdot,\theta)\Big]\,dx. \tag{2.8}$$

3. In view of the aforesaid, the computational algorithm is derived as follows:

1. The interval $[0,1]$ is divided by a uniform grid of nodes $\{x_n\}$ into a rather large number of segments of small length, which model the set of small measure D_i.

2. For the permissible control $[\theta(x), l]$, we obtain the natural frequency by solving Eq. (1.7) by the Muller method [5].

3. We solve system (2.2) for the frequency obtained, assuming that on the segment D_i the value of the vector of state variables is characterized by its value in the middle of the segment $x = x_i + h/2$, where h is the length of the small segment.

4. On the segment D_i, we specify a new control $\tilde{\theta}$ from the condition (2.8).

5. If the new control coincides with the old control, we revert to item 2, increasing the subscript of the small segment by unity. Otherwise, from condition (2.3), we calculate δl_i, assuming that $F_1 + \delta F_1 \approx 0$:

$$\delta l_i = \Big\{P_* - \eta\int_0^1 (R-l)[l + x(R-l)]^2\rho(x)\,dx - \eta\int_D (R-l)r^2[\tilde{\rho}(x) - \rho(x)]\,dx\Big\}(\eta B_1)^{-1}. \tag{3.1}$$

In this case, it is necessary to take into account that the measure of the segments D_i and $|\delta l|$ should be small enough to ensure the use of the linear approximation (3.1).

6. With the new control $[\tilde{\theta}(x), l_i + \delta l_i]$, we revert to item 1 and consider the segment D_{i+1}.

As an example, we solve the following problem. Let a set of five viscoelastic materials be specified. The dimensionless characteristics of the materials are listed in Table 1.

It is required to design a spherical shell with maximum damping of natural oscillations whose weight is 3.5 times smaller than that of a homogeneous shell of the densest material. The initial approximation for the control is chosen in the form $[\theta(x), l] = [1; 0.8]$, which corresponds to a homogeneous sphere with inside radius $l = 0.8$ made of material No. 1 (see Table 1). This sphere has a weight of 0.65 and a natural frequency

TABLE 1

Material	ρ	E	ν	α	$\bar{a}$	β
1	4.00	90	0.25	0.4	1	0.05
2	2.86	50	0.25	0.2	1	0.05
3	1.75	30	0.25	0.2	1	0.05
4	1.00	15	0.25	0.2	1	0.05
5	2.90	65	0.25	0.3	1	0.05

of $6.687 - 3.29i$. Optimization yields a four-layer sphere consisting of alternating materials Nos. 2 and 3 (see Table 1), with inside radius $l = 0.92587$ and weight 0.1857. The sequence of the layers is as follows: material No. 2 on $[l, R_1]$, material No. 3 on $[R_1, R_2]$, material No. 2 on $[R_2, R_3]$, and material No. 3 on $[R_3, 1]$. Here $R_1 = 0.9718$, $R_2 = 0.9778$, and $R_3 = 0.99703$ (see Fig. 1).

This sphere has a frequency $2.64 - 7.46\,i$, i.e., its damping properties are improved by more than a factor of 2.

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